

# CRITERIA FOR STRICT MONOTONICITY OF THE MIXED VOLUME OF CONVEX POLYTOPES

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ABSTRACT. Let  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  be convex polytopes in  $\mathbb{R}^n$  such that  $P_i \subset Q_i$ . We give criteria describing when the mixed volume is strictly increasing

$$V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n).$$

This geometric result allows us to characterize sparse polynomial systems with Newton polytopes  $P_1, \dots, P_n$  whose number of isolated solutions equals the normalized volume of the convex hull of  $P_1 \cup \dots \cup P_n$ . In addition, we obtain an analog of Cramer's rule for sparse polynomial systems.

## 1. INTRODUCTION

The mixed volume is one of the fundamental notions in the theory of convex bodies. It plays a central role in the Brunn–Minkowski theory and in the theory of sparse polynomial systems. The mixed volume is the polarization of the volume form on the space of convex bodies in  $\mathbb{R}^n$ . More precisely, let  $K_1, \dots, K_n$  be  $n$  convex bodies in  $\mathbb{R}^n$  and  $\text{Vol}_n(K)$  the Euclidean volume of a body  $K \subset \mathbb{R}^n$ . Then the mixed volume of  $K_1, \dots, K_n$  is

$$(1.1) \quad V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{m=1}^n (-1)^{n+m} \sum_{i_1 < \dots < i_m} \text{Vol}_n(K_{i_1} + \dots + K_{i_m}),$$

where  $K + L = \{x + y \in \mathbb{R}^n \mid x \in K, y \in L\}$  denotes the Minkowski sum of bodies  $K$  and  $L$ . It is not hard to see that the mixed volume is symmetric and multilinear with respect to Minkowski addition. Also it coincides with the volume on the diagonal, i.e.  $V(K, \dots, K) = \text{Vol}_n(K)$  and is invariant under translations. Moreover, it satisfies the following *monotonicity property*, which is not apparent from the definition, see [8, (5.25)]. If  $L_1, \dots, L_n$  are convex bodies such that  $K_i \subseteq L_i$  for  $1 \leq i \leq n$  then

$$V(K_1, \dots, K_n) \leq V(L_1, \dots, L_n).$$

The main goal of this paper is to give a geometric criterion for strict monotonicity in the class of convex polytopes. We give two equivalent criteria in terms of essential collections of faces and mixed cells in mixed polyhedral subdivisions, see Theorem 3.3 and Theorem 4.1. The criterion is especially simple when all  $L_i$  are equal (Corollary 3.6) which is the situation in our application to sparse polynomial systems. In the general case of convex bodies this is still an open problem, see [8, pp. 429–431] for special cases and conjectures.

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The role of mixed volumes in algebraic geometry originates in the work of Bernstein, Kushnirenko, and Khovanskii, who gave a vast generalization of the classical Bézout formula for the intersection number of hypersurfaces in the projective space, see [1, 5, 4]. This beautiful result which links algebraic geometry and convex geometry through toric varieties and sparse polynomial systems is commonly known as the BKK bound. In particular, it says that if  $f_1(x) = \cdots = f_n(x) = 0$  is an  $n$ -variate Laurent polynomial system over an algebraically closed field  $\mathbb{K}$  then the number of its isolated solutions in the algebraic torus  $(\mathbb{K}^*)^n$  is at most  $n!V(P_1, \dots, P_n)$ , where  $P_i$  are the Newton polytopes of the  $f_i$ . (Here  $\mathbb{K}^*$  denotes  $\mathbb{K} \setminus \{0\}$ .) Systems that have precisely  $n!V(P_1, \dots, P_n)$  solutions in  $(\mathbb{K}^*)^n$  must satisfy a *non-degeneracy condition* which means that certain subsystems have to be inconsistent, see Theorem 5.1.

Let  $f_1(x) = \cdots = f_n(x) = 0$  be a Laurent polynomial system over  $\mathbb{K}$  with Newton polytopes  $P_1, \dots, P_n$ . Replacing each  $f_i$  with a generic linear combination of  $f_1, \dots, f_n$  over  $\mathbb{K}$  produces an equivalent system with the same number of solutions in  $(\mathbb{K}^*)^n$ . Such an operation replaces each individual Newton polytope  $P_i$  with the convex-hull of their union,  $Q = \text{conv}(P_1 \cup \cdots \cup P_n)$ . Thus, starting with a system for which  $V(P_1, \dots, P_n) < V(Q, \dots, Q) = \text{Vol}_n(Q)$ , one obtains a system with all Newton polytopes equal to  $Q$  and which has less than  $n! \text{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$ , i.e. is degenerate. The geometric criterion of Corollary 3.6 allows us to characterize such systems without checking the non-degeneracy condition, which could be hard. In fact, Theorem 5.5 delivers a simple characterization in terms of the coefficient matrix  $C$  and the augmented exponent matrix  $\bar{A}$  of the system (see Section 5 for definitions). In particular, it says that if  $Q$  has a proper face such that the rank of the corresponding submatrix of  $C$  is less than the rank of the corresponding submatrix of  $\bar{A}$  then the system has less than  $n! \text{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$ .

Here is another consequence of Theorem 5.5. If no maximal minor of  $C$  vanishes then the system has the maximal number  $n! \text{Vol}_n(Q)$  of isolated solutions in  $(\mathbb{K}^*)^n$  (Corollary 5.7). This can be thought of as a generalization of Cramer's rule for linear systems.

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## 2. PRELIMINARIES

In this section we recall necessary definitions and results from convex geometry and set up notation. In addition, we recall the notion of essential collections of polytopes for which we give several equivalent definitions, as well as define mixed polyhedral subdivisions and the combinatorial Cayley trick.

Throughout the paper we use  $[n]$  to denote the set  $\{1, \dots, n\}$ .

**Mixed Volume.** For a convex body  $K$  in  $\mathbb{R}^n$  the function  $h_K : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ , given by  $h_K(u) = \max\{\langle u, x \rangle \mid x \in K\}$  is the *support function* of  $K$ . We sometimes enlarge the domain of  $h_K$  to  $(\mathbb{R}^n)^* = \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ . For every  $u \in (\mathbb{R}^n)^*$ , we write  $H_K(u)$

to denote the supporting hyperplane for  $K$  with outer normal  $u$

$$H_K(u) = \{x \in \mathbb{R}^n : \langle u, x \rangle = h_K(u)\}.$$

We use  $K^u$  to denote the face  $K \cap H_K(u)$  of  $K$ .

Let  $V(K_1, \dots, K_n)$  be the  $n$ -dimensional mixed volume of  $n$  convex bodies  $K_1, \dots, K_n$  in  $\mathbb{R}^n$ , see (1.1) above. We have the following equivalent definition.

**Theorem 2.1.** [8, Theorem 5.1.7] *Let  $\lambda_1, \dots, \lambda_n$  be non-negative real numbers. Then  $\text{Vol}_n(\lambda_1 K_1 + \dots + \lambda_n K_n)$  is a polynomial in  $\lambda_1, \dots, \lambda_n$  whose coefficient of the monomial  $\lambda_1 \dots \lambda_n$  equals  $V(K_1, \dots, K_n)$ .*

**Essential Collections.** Let  $K_1, \dots, K_m$  be convex bodies in  $\mathbb{R}^n$ , not necessarily distinct. We say that a multiset  $\{K_1, \dots, K_m\}$  is an *essential collection* if for any subset  $I \subset [m]$  of size at most  $n$  we have

$$\dim \sum_{i \in I} K_i \geq |I|.$$

Note that every sub-collection of an essential collection is essential. Also  $\{K, \dots, K\}$ , where  $K$  is repeated  $m$  times, is essential if and only if  $\dim K \geq m$ .

The following is a well-known property of essential collections.

**Theorem 2.2.** [8, Theorem 5.1.8] *Let  $K_1, \dots, K_n$  be  $n$  convex bodies in  $\mathbb{R}^n$ . The following are equivalent:*

- (1)  $V(K_1, \dots, K_n) > 0$ ;
- (2) *There exist segments  $E_i \subset K_i$  for  $1 \leq i \leq n$  with linearly independent directions;*
- (3)  $\{K_1, \dots, K_n\}$  *is essential.*

Another useful result is the inductive formula for the mixed volume, see [8, Theorem 5.1.7, (5.19)]. We present a variation of this formula for convex polytopes. Let  $K$  be a convex body and  $Q_2, \dots, Q_n$  convex polytopes in  $\mathbb{R}^n$ . Given  $u \in \mathbb{S}^{n-1}$ , let  $V(Q_2^u, \dots, Q_n^u)$  denote the  $(n-1)$ -dimensional mixed volume of  $Q_2^u, \dots, Q_n^u$  translated to the orthogonal subspace  $u^\perp$ . Then we have

$$(2.1) \quad V(K, Q_2, \dots, Q_n) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_K(u) V(Q_2^u, \dots, Q_n^u).$$

Note that the above sum is finite, since there are only finitely many  $u \in \mathbb{S}^{n-1}$  for which  $\{Q_2^u, \dots, Q_n^u\}$  is essential. Namely, these  $u$  are among the outer unit normals to the facets of  $Q_2 + \dots + Q_n$ .

**Combinatorial Cayley Trick.** Let  $P_1, \dots, P_k \subset \mathbb{R}^n$  be convex polytopes. The *Cayley polytope*  $\mathcal{C}(P_1, \dots, P_k)$  is the convex hull in  $\mathbb{R}^n \times \mathbb{R}^k$  of the union of the polytopes  $P_i \times \{e_i\}$  for  $i = 1, \dots, k$ , where  $\{e_1, \dots, e_k\}$  is the standard basis for  $\mathbb{R}^k$ .

A finite polyhedral subdivision of  $P_1 + \dots + P_k$  is called *mixed* if it corresponds to a polyhedral subdivision of  $\mathcal{C}(P_1, \dots, P_k)$  with vertices in  $\cup_{i=1}^k P_i \times \{e_i\}$  via the map sending a polytope  $\mathcal{C}$  of the subdivision of  $\mathcal{C}(P_1, \dots, P_k)$  to the polytope  $\sigma = \sigma_1 + \dots + \sigma_k$ , where  $\sigma_i \subset P_i$  is the image by the projection  $\mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  of  $\mathcal{C} \cap \{(x_1, \dots, x_n, y_1, \dots, y_k) \in \mathbb{R}^n \times \mathbb{R}^k \mid y_i = 1\}$ . The polytope  $\mathcal{C}$  is determined by  $\sigma$  (and

vice versa) and will be denoted  $\mathcal{C}_\sigma$ . This correspondence is called the *combinatorial Cayley trick*, see [6] for instance.

A mixed polyhedral subdivision of  $P_1 + \dots + P_k$  is called *pure* if the corresponding subdivision of  $\mathcal{C}(P_1, \dots, P_k)$  is a triangulation. Let  $\sigma = \sigma_1 + \dots + \sigma_k$  be a polytope in a pure mixed polyhedral subdivision of  $P_1 + \dots + P_k$ . Then all  $\sigma_i$  are simplices and  $\dim(\sigma_1 + \dots + \sigma_k) = \dim \sigma_1 + \dots + \dim \sigma_k$ . The polytope  $\sigma$ , as well as the corresponding simplex  $\mathcal{C}_\sigma$  of the triangulation of  $\mathcal{C}(P_1, \dots, P_k)$ , is called *fully mixed* if  $\dim \sigma_i \geq 1$  for  $1 \leq i \leq k$ . When  $k = n$ , the polytope  $\sigma$  is fully mixed if and only if  $\dim \sigma = n$  and  $\dim \sigma_i = 1$  for  $1 \leq i \leq k$  so that  $\sigma$  is an  $n$ -dimensional parallelotope.

The following result is well-known, see [7, Theorem 2.4] or [2, Theorem 6.7]. We include a proof for reader's convenience.

**Lemma 2.3.** *For convex polytopes  $P_1, \dots, P_n$  in  $\mathbb{R}^n$ , the quantity  $n!V(P_1, \dots, P_n)$  is equal to the sum of the Euclidean volumes of the fully mixed polytopes in any pure mixed polyhedral subdivision of  $P_1 + \dots + P_n$ .*

*Proof.* Consider a triangulation  $\tau$  of  $\mathcal{C}(P_1, \dots, P_n)$  and let  $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}_{\geq 0}^n$ . To any simplex  $\mathcal{C}_\sigma = \text{conv}(\cup_{i=1}^n \sigma_i \times \{e_i\})$  of  $\tau$ , we associate the simplex  $\text{conv}(\cup_{i=1}^n (\lambda_i \sigma_i) \times \{e_i\})$ . The image of this map is a triangulation of  $\mathcal{C}(\lambda_1 P_1, \dots, \lambda_n P_n)$ . Then we may compute the Euclidean volume of  $\lambda_1 P_1 + \dots + \lambda_n P_n$  as the sum of the Euclidean volumes of the polytopes in the corresponding pure mixed subdivision:

$$\begin{aligned} \text{Vol}_n(\lambda_1 P_1 + \dots + \lambda_n P_n) &= \sum_{\mathcal{C}_\sigma \in \tau} \text{Vol}_n(\lambda_1 \sigma_1 + \dots + \lambda_n \sigma_n) \\ &= \sum_{\mathcal{C}_\sigma \in \tau} \text{Vol}_n(\sigma_1 + \dots + \sigma_n) \cdot \lambda_1^{\dim(\sigma_1)} \dots \lambda_n^{\dim(\sigma_n)}, \end{aligned}$$

the latter equality coming from the fact that  $\tau$  is a triangulation. The coefficient of  $\lambda_1 \dots \lambda_n$  in the last expression is precisely the total Euclidean volume of the fully mixed polytopes in our fully mixed subdivision, and by Theorem 2.1 coincides with the mixed volume of  $P_1, \dots, P_n$ .  $\square$

### 3. FIRST CRITERION

In this section we present our first criterion for strict monotonicity of the mixed volume and its corollaries.

**Definition 3.1.** Let  $K$  be a subset of a convex polytope  $A$  and let  $F \subset A$  be a facet. We say  $K$  *touches*  $F$  when the intersection  $K \cap F$  is non-empty.

We will often make use of the following proposition, which gives a criterion for strict monotonicity in a very special case, see [8, page 282].

**Proposition 3.2.** *Let  $P_1, Q_1, \dots, Q_n$  be convex polytopes in  $\mathbb{R}^n$  and  $P_1 \subseteq Q_1$ . Then  $V(P_1, Q_2, \dots, Q_n) = V(Q_1, Q_2, \dots, Q_n)$  if and only if  $P_1$  touches every face  $Q_1^u$  for  $u$  in the set*

$$U = \{u \in \mathbb{S}^{n-1} \mid \{Q_2^u, \dots, Q_n^u\} \text{ is essential}\}.$$

The above statement easily follows from (2.1) and the observation  $h_{P_1}(u) \leq h_{Q_1}(u)$  with equality if and only if  $P_1$  touches  $Q_1^u$ . See [8, Sec 5.1] for details.

Here is the first criterion for strict monotonicity.

**Theorem 3.3.** *Let  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  be convex polytopes in  $\mathbb{R}^n$  such that  $P_i \subseteq Q_i$  for every  $i \in [n]$ . Given  $u \in \mathbb{S}^{n-1}$  consider the set*

$$T_u = \{i \in [n] \mid P_i \text{ touches } Q_i^u\}.$$

*Then  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$  if and only if there exists  $u \in \mathbb{S}^{n-1}$  such that the collection  $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$  is essential.*

*Proof.* Assume that there exists  $u \in \mathbb{S}^{n-1}$  such that the collection  $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$  is essential. Note that  $T_u$  is a proper subset of  $[n]$ , otherwise  $\{Q_i^u \mid i \in T_u\}$  is a collection of  $n$  polytopes contained in translates of an  $(n-1)$ -dimensional subspace, hence, cannot be essential. Without loss of generality we may assume that  $[n] \setminus T_u = \{1, \dots, k\}$  for some  $k \geq 1$ . Since  $P_i$  does not touch  $Q_i^u$  for  $1 \leq i \leq k$  there is a hyperplane  $H = \{x \in \mathbb{R}^n \mid \langle x, u \rangle = h_{Q_i}(u) - \varepsilon\}$  which separates  $P_i$  and  $Q_i^u$ . Let  $H_+$  be the half-space containing  $P_i$ . Then the truncated polytope  $\tilde{Q}_i = Q_i \cap H_+$  satisfies  $P_i \subseteq \tilde{Q}_i \subset Q_i$ . We claim that the collection

$$(3.1) \quad \{\tilde{Q}_2^u, \dots, \tilde{Q}_k^u, Q_{k+1}^u, \dots, Q_n^u\}$$

is essential. Indeed, by assumption there exist  $n$  segments  $E_i \subset Q_i$  with linearly independent directions such that  $E_i \subset Q_i^u$  for  $k < i \leq n$ . Among the first  $k$  of these segments no more than one can be parallel to  $u$ , hence, by projecting them along  $u$ , translating, and possibly reordering, we may assume that  $E_i \subset \tilde{Q}_i^u$  for  $2 \leq i \leq k$ . Since the direction vectors of  $E_2, \dots, E_n$  are linearly independent in the orthogonal subspace  $u^\perp$ , the collection in (3.1) is essential.

Now by Proposition 3.2 we obtain

$$V(P_1, \tilde{Q}_2, \dots, \tilde{Q}_k, Q_{k+1}, \dots, Q_n) < V(Q_1, \tilde{Q}_2, \dots, \tilde{Q}_k, Q_{k+1}, \dots, Q_n).$$

Finally, by monotonicity we have  $V(P_1, \dots, P_n) \leq V(P_1, \tilde{Q}_2, \dots, \tilde{Q}_k, Q_{k+1}, \dots, Q_n)$  and  $V(Q_1, \tilde{Q}_2, \dots, \tilde{Q}_k, Q_{k+1}, \dots, Q_n) \leq V(Q_1, \dots, Q_n)$ . Therefore,

$$V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n).$$

Conversely, assume  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$ . Then, by monotonicity, for some  $1 \leq k \leq n$  we have

$$V(P_1, \dots, P_{k-1}, P_k, Q_{k+1}, \dots, Q_n) < V(P_1, \dots, P_{k-1}, Q_k, Q_{k+1}, \dots, Q_n).$$

By Proposition 3.2 there exists  $u \in \mathbb{S}^{n-1}$  such that  $\{P_1^u, \dots, P_{k-1}^u, Q_{k+1}^u, \dots, Q_n^u\}$  is essential and  $k \notin T_u$ . By choosing a segment in  $Q_k$  not parallel to  $u^\perp$  (which exists since  $P_k \subset Q_k$ , but  $P_k$  does not touch  $Q_k^u$ ) we see that  $\{P_1^u, \dots, P_{k-1}^u, Q_k, Q_{k+1}^u, \dots, Q_n^u\}$  is essential. It remains to notice that  $P_i^u \subseteq Q_i^u$  for  $i \in T_u$  and, hence, the collection  $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$  is essential as well.  $\square$

**Remark 3.4.** Note that if  $Q_1, \dots, Q_n$  are  $n$ -dimensional then we can simplify the criterion of Theorem 3.3 as follows:  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$  if and only if there exists  $u \in \mathbb{S}^{n-1}$  such that the collection  $\{Q_i^u \mid i \in T_u\}$  is essential.

**Example 3.5.** For  $n = 2$  Theorem 3.3 says that  $V(P_1, P_2) < V(Q_1, Q_2)$  if and only if, up to reordering, there exists a facet (i.e. a side)  $Q_1^u$  such that the corresponding face  $Q_2^u$  is not touched by  $P_2$ .

A particular instance of Theorem 3.3, especially important for applications to polynomial systems, is the case when  $P_1, \dots, P_n$  are arbitrary polytopes and  $Q_1 = \dots = Q_n = Q$ , where  $Q$  is the convex hull of the union  $P_1 \cup \dots \cup P_n$ . We will assume that  $Q$  is  $n$ -dimensional, otherwise  $\{P_1, \dots, P_n\}$  is not essential and, hence, both  $V(P_1, \dots, P_n)$  and  $V(Q, \dots, Q) = \text{Vol}_n(Q)$  are zero. Then the strict monotonicity has the following simple geometric interpretation.

**Corollary 3.6.** *Let  $P_1, \dots, P_n$  be polytopes in  $\mathbb{R}^n$  contained in an  $n$ -dimensional polytope  $Q$ . Then  $V(P_1, \dots, P_n) < \text{Vol}_n(Q)$  if and only if for some  $s \in [n]$  there exist  $s$  polytopes among  $P_1, \dots, P_n$  that do not touch a codimension  $s$  face of  $Q$ .*

*Proof.* By Theorem 3.3 and Remark 3.4, we have  $V(P_1, \dots, P_n) < \text{Vol}_n(Q)$  if and only if there exists  $u \in \mathbb{S}^{n-1}$  such that the collection  $\{Q^u, \dots, Q^u\}$ , where  $Q^u$  is repeated  $|T_u|$  times, is essential. The last condition is equivalent to  $\dim Q^u \geq |T_u|$ , which gives  $n - |T_u| \geq s$  where  $s$  is the codimension of  $Q^u$ . This precisely means that there exist  $s$  polytopes among the  $P_i$  which do not touch  $Q^u$ .  $\square$

**Remark 3.7.** Corollary 3.6 can be reformulated as follows. Let  $P_1, \dots, P_n$  be polytopes in  $\mathbb{R}^n$  contained in an  $n$ -dimensional polytope  $Q$ . Then  $V(P_1, \dots, P_n) < \text{Vol}_n(Q)$  if and only if there is a proper face of  $Q$  of dimension  $t$  which is touched by at most  $t$  of the polytopes  $P_1, \dots, P_n$ . In particular, if a vertex of  $Q$  does not belong to any of the polytopes  $P_1, \dots, P_n$ , then  $V(P_1, \dots, P_n) < \text{Vol}_n(Q)$ .

**Example 3.8.** Let  $P_1, P_2$  be convex polytopes in  $\mathbb{R}^2$  and  $Q$  be the convex hull of their union. Then Corollary 3.6 shows that  $V(P_1, P_2) < V(Q)$  if and only if either  $P_1$  or  $P_2$  does not touch some side of  $Q$ .

**Remark 3.9.** One can obtain a more direct proof of Corollary 3.6 by modifying the proof of Theorem 2.6 in [9].

**Example 3.10.** Let  $\mathcal{A}$  be a finite set in  $\mathbb{R}^n$  with  $n$ -dimensional convex hull  $Q$  and choose a subset  $\{a_1, \dots, a_n\} \subset \mathcal{A}$ . Define  $\mathcal{A}_i = (\mathcal{A} \setminus \{a_1, \dots, a_n\}) \cup \{a_i\}$  for  $1 \leq i \leq n$  and let  $P_i$  be the convex hull of  $\mathcal{A}_i$ . Then Corollary 3.6 leads to  $V(P_1, \dots, P_n) = \text{Vol}_n(Q)$ . Indeed, if  $I \subset [n]$  has size  $s$  and  $F \subset Q$  is a face of codimension  $s$ , then  $|\cup_{i \in I} \mathcal{A}_i| = |\mathcal{A}| - n + s$  and  $|F \cap \mathcal{A}| \geq n - s + 1$ . Therefore, the subsets  $\cup_{i \in I} \mathcal{A}_i$  and  $F \cap \mathcal{A}$  cannot be disjoint, i.e. the union of any  $s$  of the  $P_i$  touches every codimension face of  $Q$ .

Going back to the statement of Corollary 3.6 it is natural to ask: If there is a codimension  $s$  face of  $Q$  not touched by more than  $s$  polytopes among  $P_1, \dots, P_n$ , can the inequality  $V(P_1, \dots, P_n) < \text{Vol}_n(Q)$  be improved? The answer is clearly no if we do not restrict ourselves to the class of *lattice polytopes*. Recall that for any collection of lattice polytopes  $P_1, \dots, P_n$  the normalized mixed volume  $n! V(P_1, \dots, P_n)$  is an integer. We have the following improvement of Corollary 3.6 in this case.

**Proposition 3.11.** *Let  $P_1, \dots, P_n$  be lattice polytopes in  $\mathbb{R}^n$  contained in an  $n$ -dimensional lattice polytope  $Q$ . Suppose there exists a facet  $Q^u \subset Q$  which is not touched by  $P_1, \dots, P_m$ , for some  $m \geq 1$ . Moreover, suppose that  $\{P_1^u, \dots, P_{m-1}^u\}$ . Then  $n! V(P_1, \dots, P_n) \leq n! \text{Vol}_n(Q) - m$ .*

*Proof.* By the essentiality of  $\{P_1^u, \dots, P_{m-1}^u\}$  and since  $\dim Q^u = n - 1$  it follows that for any  $0 \leq i \leq m - 1$  the collection  $\{P_1^u, \dots, P_i^u, Q^u, \dots, Q^u\}$ , where  $Q^u$  is repeated  $n - i - 1$  times, is essential. By Proposition 3.2 we obtain

$$(3.2) \quad V(P_1, \dots, P_i, \underbrace{Q, \dots, Q}_{n-i}) > V(P_1, \dots, P_{i+1}, \underbrace{Q, \dots, Q}_{n-i-1}).$$

Now to obtain  $n!V(P_1, \dots, P_n) \leq n! \operatorname{Vol}_n(Q) - m$  we use (3.2) successively with  $i = 0, \dots, m - 1$ :

$$\operatorname{Vol}_n(Q) > V(P_1, Q, \dots, Q) > V(P_1, P_2, Q, \dots, Q) > \dots > V(P_1, \dots, P_m, Q, \dots, Q).$$

Multiplying each of the above terms by  $n!$  we get a decreasing sequence of  $m + 1$  integers, the last being no less than  $n!V(P_1, \dots, P_n)$ , which provides the required inequality.  $\square$

We remark that the essentiality conditions above cannot be removed. Indeed, let  $Q = \operatorname{conv}\{0, e_1, \dots, e_n\}$  be the standard  $n$ -simplex,  $Q^u$  one of its facets, and  $P_i \subset Q$  for  $1 \leq i \leq n$ . Then if  $P_1, \dots, P_m$  equal the vertex of  $Q$  not contained in  $Q^u$ , then

$$0 = n!V(P_1, \dots, P_n) = n!V_n(Q) - 1,$$

regardless of  $m$ . It would be interesting to obtain a more general statement than Proposition 3.11 which deals with faces of larger codimension.

**Remark 3.12.** Given a polytope  $Q$ , Corollary 3.6 provides a characterization of collections  $P_1, \dots, P_n$  such that  $P_i \subset Q$  for  $1 \leq i \leq n$  and  $V(P_1, \dots, P_n) = \operatorname{Vol}_n(Q)$ . Clearly, when  $Q$  and the  $P_i$  are lattice polytopes there are only finitely many such collections. Describing them explicitly is a hard combinatorial problem. In the case when  $Q$  is the standard simplex,  $Q = \operatorname{conv}\{0, e_1, \dots, e_n\}$ , this problem was solved by Esterov and Gusev in [3].

#### 4. SECOND CRITERION

In this section we obtain another criterion for the strict monotonicity property (Theorem 4.1) based on mixed polyhedral subdivisions and the combinatorial Cayley trick.

Recall that for polytopes  $Q_1, \dots, Q_n$  in  $\mathbb{R}^n$ , we have  $V(Q_1, \dots, Q_n) > 0$  if and only if the collection  $\{Q_1, \dots, Q_n\}$  is essential, which is equivalent to  $\dim \sum_{i \in I} Q_i \geq |I|$  for all  $I \subset [n]$ , see Theorem 2.2. We now describe a generalization of this criterion. Consider again polytopes  $P_i \subset Q_i \subset \mathbb{R}^n$  for  $i = 1, \dots, n$ . For any non-zero vector  $u \in \mathbb{R}^n$  consider the convex polytopes

$$B_{i,u} = \{x \in Q_i \mid \langle u, x \rangle \geq h_{P_i}(u)\}, \quad i = 1, \dots, n.$$

**Theorem 4.1.** *Let  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  be convex polytopes in  $\mathbb{R}^n$  such that  $P_i \subseteq Q_i$  for every  $i \in [n]$ . The following conditions are equivalent:*

- (1)  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$ ,
- (2) *there exists a fully mixed simplex with vertices in  $\cup_{i=1}^n Q_i \times \{e_i\}$  which is contained in the closure of  $\mathcal{C}(Q_1, \dots, Q_n) \setminus \mathcal{C}(P_1, \dots, P_n)$ ,*
- (3) *there exists a non-zero vector  $u \in \mathbb{R}^n$  such that the collection  $\{B_{1,u}, \dots, B_{n,u}\}$  is essential.*

*Proof.* Consider a fully mixed simplex  $\mathcal{C}_E \subset \mathcal{C}(Q_1, \dots, Q_n)$  and let  $E = E_1 + \dots + E_n$  be the corresponding fully mixed polytope in the mixed subdivision of  $P = P_1 + \dots + P_n$ . Then  $\mathcal{C}_E$  is contained in the closure of  $\mathcal{C}(Q_1, \dots, Q_n) \setminus \mathcal{C}(P_1, \dots, P_n)$  if and only if there is a supporting hyperplane of  $\mathcal{C}(P_1, \dots, P_n)$  which separates the convex sets  $\mathcal{C}(P_1, \dots, P_n)$  and  $\mathcal{C}_E$ . This is equivalent to the existence of  $(u, v) \in \mathbb{R}^{2n}$  with  $u \neq 0$  such that

$$(4.1) \quad \mathcal{C}_E \subset \{(x, y) \in \mathbb{R}^{2n} \mid \langle u, x \rangle + \langle v, y \rangle \geq h_{\mathcal{C}(P_1, \dots, P_n)}(u, v)\}.$$

We note that (4.1) is equivalent to  $E_i \subset \{x \in \mathbb{R}^n \mid \langle u, x \rangle \geq h_{P_i}(u)\}$ , and so  $E_i \subset B_{i,u}$ , for  $i = 1, \dots, n$ . Since  $E_1, \dots, E_n$  have linearly independent directions, this gives that  $\{B_{1,u}, \dots, B_{n,u}\}$  is essential. This shows the equivalence between (2) and (3).

Assuming that  $\mathcal{C}_E$  is a fully mixed simplex verifying (4.1), it is easy to show the existence of a triangulation of  $\mathcal{C}(Q_1, \dots, Q_n)$  with vertices in  $\cup_{i=1}^n Q_i \times \{e_i\}$  which contains  $\mathcal{C}_E$  and restricts to a triangulation of  $\mathcal{C}(P_1, \dots, P_n)$ . By the combinatorial Cayley trick, this gives a pure mixed subdivision  $\tau_Q$  of  $Q_1 + \dots + Q_n$  restricting to a pure mixed subdivision  $\tau_P$  of  $P_1 + \dots + P_n$  and a fully mixed polytope  $E$  contained in  $\tau_Q \setminus \tau_P$ . Therefore,  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$  by Lemma 2.3. The implication (2)  $\Rightarrow$  (1) is proved.

Assume now that  $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$ . By Proposition 3.2 there exists  $u \in \mathbb{S}^{n-1}$  such that  $\{P_1^u, \dots, P_{k-1}^u, Q_{k+1}^u, \dots, Q_n^u\}$  is essential and  $k \notin T_u$  (see the proof of Theorem 3.3). Thus we may choose a segment  $E_k \subset B_{k,u}$ , segments  $E_i \subset P_i^u$  for  $i = 1, \dots, k-1$ , and segments  $E_i \subset Q_i^u$  for  $i = k+1, \dots, n$  such that the corresponding  $n$  direction vectors are linearly independent. This shows that  $\{B_{1,u}, \dots, B_{n,u}\}$  is essential since  $P_i^u$  and  $Q_i^u$  are contained in  $B_{i,u}$  for  $i = 1, \dots, n$ . We have proved the implication (1)  $\Rightarrow$  (2).  $\square$

**Remark 4.2.** Note that if  $P_i$  touches  $Q_i^u$  then  $B_{i,u} = Q_i^u$  and if  $P_i$  does not touch  $Q_i^u$  then  $\dim B_{i,u} = \dim Q_i$ . Therefore, the condition (3) in the above theorem is equivalent to the condition in Theorem 3.3.

## 5. POLYNOMIAL SYSTEMS

Consider a finite set  $\mathcal{A} = \{a_1, \dots, a_\ell\} \subset \mathbb{Z}^n$  where  $\ell = |\mathcal{A}|$ . Let  $(a_{1j}, \dots, a_{nj})$  be the coordinates of  $a_j$  for  $j = 1, \dots, \ell$ . Consider a Laurent polynomial system with coefficients in an algebraically closed field  $\mathbb{K}$

$$(5.1) \quad f_1(x) = \dots = f_n(x) = 0,$$

where  $f_i(x) = \sum_{j=1}^{\ell} c_{ij} x^{a_j}$  and as usual  $x^{a_j}$  stands for the monomial  $x_1^{a_{1j}} \dots x_n^{a_{nj}}$ . We shall assume that no polynomial  $f_i$  is the null polynomial. Call  $\mathcal{A}_i = \{a_j \in \mathbb{Z}^n, c_{ij} \neq 0\}$  the *individual support* of  $f_i$  and  $\mathcal{A} = \cup_{i=1}^n \mathcal{A}_i$  the *total support* of the system (5.1). The *Newton polytope*  $P_i$  of  $f_i$  is the convex hull of  $\mathcal{A}_i$  and the *Newton polytope*  $Q$  of the system (5.1) is the convex hull of  $\mathcal{A}$ .

The matrices

$$C = (c_{ij}) \in \mathbb{K}^{n \times \ell} \text{ and } A = (a_{ij}) \in \mathbb{Z}^{n \times \ell}$$

are the *coefficient* and *exponent* matrices of the system, respectively.



Choose  $u \in \mathbb{S}^{n-1}$  and let  $\mathcal{A}_i^u = P_i^u \cap \mathcal{A}_i$ . Then the *restricted system* corresponding to  $u$  is the system

$$f_1^u(x) = \cdots = f_n^u(x) = 0,$$

where  $f_i^u(x) = \sum_{j=1}^{\ell} c_{ij}^u x^{a_j}$  with  $c_{ij}^u = c_{ij}$  if  $a_j \in \mathcal{A}_i^u$  and  $c_{ij}^u = 0$ , otherwise. Finally, a system (5.1) is called *non-degenerate* if for every  $u \in \mathbb{S}^{n-1}$  the corresponding restricted system is inconsistent.

The relation between mixed volumes and polynomial systems originates in the following fundamental result, known as the BKK bound, discovered by Bernstein, Kushnirenko, and Khovanskii, see [1, 5, 4].

**Theorem 5.1.** *The system (5.1) has at most  $n!V(P_1, \dots, P_n)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity. Moreover, it has precisely  $n!V(P_1, \dots, P_n)$  solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity if and only if it is non-degenerate.*

**Remark 5.2.** Systems with fixed individual supports and generic coefficients are non-degenerate. Moreover, the non-degeneracy condition is not needed if one passes to the toric compactification  $X_P$  associated with the polytope  $P = P_1 + \cdots + P_n$ . Namely, a system has at most  $n!V(P_1, \dots, P_n)$  isolated solutions in  $X_P$  counted with multiplicity, and if it has a finite number of solutions in  $X_P$  then this number equals  $n!V(P_1, \dots, P_n)$  counted with multiplicity.

There are two operations on (5.1) that preserve its number of solutions in the torus  $(\mathbb{K}^*)^n$ . First, left multiplication of  $C$  by an invertible matrix of  $M_n(\mathbb{K})$  produces an equivalent system. The second operation consists in multiplying each equation by a given monomial and making a monomial change of coordinates of  $(\mathbb{K}^*)^n$ . This second operation corresponds to left multiplication of the *augmented exponent matrix*

$$\bar{A} = \begin{pmatrix} 1 & \cdots & 1 \\ & A & \end{pmatrix} \in \mathbb{Z}^{(n+1) \times \ell}$$

(obtained from  $A$  by adding a first row of 1) by a matrix in  $\mathrm{GL}_{n+1}(\mathbb{Z})$ .

**Remark 5.3.** Consider a non-degenerate system with coefficient matrix  $C$ . Left multiplication of  $C$  by an invertible matrix preserves the total support of the system, but does not preserve the individual supports and Newton polytopes in general. However, since this operation produces an equivalent system, Theorem 5.1 shows that the mixed volumes of the corresponding  $n$ -tuples of Newton polytopes are equal.

**Example 5.4.** Assume that (5.1) has precisely  $n! \mathrm{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity and  $C$  has a non-zero maximal minor. Up to renumbering, we may assume that this minor is given by the first  $n$  columns of  $C$ . Left multiplication of (5.1) by the inverse of the corresponding submatrix of  $C$  gives an equivalent system with individual supports  $\mathcal{A}'_i \subset \mathcal{A}''_i = (\mathcal{A} \setminus \{a_1, \dots, a_n\}) \cup \{a_i\}$  for  $1 \leq i \leq n$ . Thus this new system has precisely  $n! \mathrm{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity. By Theorem 5.1 this number of solutions is at most  $n!V(P'_1, \dots, P'_n)$ , where  $P'_i$  is the convex hull of  $\mathcal{A}'_i$ . On the other hand, by monotonicity of the mixed volume we have  $V(P'_1, \dots, P'_n) \leq V(P''_1, \dots, P''_n) \leq \mathrm{Vol}_n(Q)$ . We conclude that  $V(P'_1, \dots, P'_n) = V(P''_1, \dots, P''_n) = \mathrm{Vol}_n(Q)$ . The second equality is also a consequence of Corollary 3.6, see Example 3.10.

**Theorem 5.5.** *Assume  $\dim Q = n$ . If a system (5.1) has  $n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity, then for any proper face  $F$  of  $Q$  the submatrix  $C_{\mathcal{F}} \in \mathbb{K}^{n \times |\mathcal{F}|}$  of  $C$  with columns indexed by  $\mathcal{F} = \{j \in [\ell], a_j \in F \cap \mathcal{A}\}$  satisfies*

$$\operatorname{rank} C_{\mathcal{F}} \geq \dim F + 1,$$

*or equivalently,*

$$(5.2) \quad \operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}.$$

*Conversely, if (5.2) is satisfied for all proper faces  $F$  of  $Q$  and if the system (5.1) is non-degenerate, then it has precisely  $n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity.*

*Proof.* First, note that for any proper face  $F$  of  $Q$  we have  $\operatorname{rank} \bar{A}_{\mathcal{F}} = \dim F + 1$ . Consider a proper face  $F$  of  $Q$  of codimension  $s \geq 1$  and assume that  $\operatorname{rank} C_{\mathcal{F}} \leq \dim F = n - s$ . Then there exist an invertible matrix  $L$  and  $I \subset [n]$  of size  $|I| = s$  such that the submatrix of  $C' = LC$  with rows indexed by  $I$  and columns indexed by  $\mathcal{F}$  is the null matrix. The matrix  $C'$  is the coefficient matrix of an equivalent system with the same total support, see Remark 5.3. Denote by  $P'_1, \dots, P'_n$  the individual Newton polytopes of this equivalent system. Then the polytopes  $P'_i$  for  $i \in I$  do not touch the face  $F$  of  $Q$ . Since  $\dim F = n - s$  and  $|I| = s$ , it follows then from Corollary 3.6 that  $V(P'_1, \dots, P'_n) < \operatorname{Vol}_n(Q)$ . Theorem 5.1 applied to the system with coefficient matrix  $C'$  gives that it has at most  $n! V(P'_1, \dots, P'_n) < n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity. The same conclusion holds for the equivalent system (5.1). Therefore, if (5.1) has  $n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity, then (5.2) is satisfied for all proper faces  $F$  of  $Q$ .

Conversely, assume that (5.1) is non-degenerate and that (5.2) is satisfied for all proper faces  $F$  of  $Q$ . Then (5.1) has  $n! V(P_1, \dots, P_n)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity by Theorem 5.1. Suppose that  $V(P_1, \dots, P_n) < \operatorname{Vol}_n(Q)$ . Then by Corollary 3.6 there exists a proper face  $F$  of  $Q$  of codimension  $s \geq 1$  and  $I \subset [n]$  of size  $|I| = s$  such that the polytopes  $P_i$  for  $i \in I$  do not touch  $F$ . But then  $\operatorname{rank} C_{\mathcal{F}} \leq n - s = \dim F$ , which gives a contradiction. Thus  $V(P_1, \dots, P_n) = \operatorname{Vol}_n(Q)$  and (5.2) has  $n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity.  $\square$

As an immediate consequence of Theorem 5.5 from which we keep the notations, we get the following.

**Corollary 5.6.** *Consider any Laurent polynomial system (5.1) with  $\dim Q = n$ . If there exists a proper face  $F$  of  $Q$  such that  $\operatorname{rank} C_{\mathcal{F}} < \operatorname{rank} \bar{A}_{\mathcal{F}}$ , then the system has either infinitely many solutions, or strictly less than  $n! \operatorname{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity.*

*Proof.* Assume the existence of a proper face  $F$  of  $Q$  such that  $\operatorname{rank} C_{\mathcal{F}} < \operatorname{rank} \bar{A}_{\mathcal{F}}$ . If (5.1) has precisely  $n! \operatorname{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity, then it is non-degenerate by Theorem 5.1 and thus  $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$  by Theorem 5.5, a contradiction.  $\square$

A very nice consequence of Theorem 5.5 is the following result, which can be considered as a generalization of Cramer's rule to polynomial systems.

**Corollary 5.7.** *Assume that  $\dim Q = n$  and that no maximal minor of  $C$  vanishes. Then the system (5.1) has the maximal number of  $n! \operatorname{Vol}_n(Q)$  isolated solutions in  $(\mathbb{K}^*)^n$  counted with multiplicity.*

*Proof.* Note that  $\ell \geq n + 1$  since  $\dim Q = n$  (recall that  $\ell$  is the number of columns of  $C$ ). Thus a maximal minor of  $C$  has size  $n$  and the fact that no maximal minor of  $C$  vanishes implies that for any  $J \subset [\ell]$  the submatrix of  $C$  with rows indexed by  $[n]$  and columns indexed by  $J$  has maximal rank. This rank is equal to  $n$  if  $|J| \geq n$  or to  $|J|$  if  $|J| < n$ . Since  $|\mathcal{F}| \geq \dim F + 1 = \operatorname{rank} \bar{A}_{\mathcal{F}}$  for any face  $F$  of  $Q$ , we get that  $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$  for any proper face  $F$  of  $Q$ . Moreover, no restricted system is consistent for otherwise this would give a non-zero vector in the kernel of the corresponding submatrix of  $C$ . Thus (5.1) is non-degenerate and the result follows from Theorem 5.5.  $\square$

When the polytope  $Q = \operatorname{conv}\{0, e_1, \dots, e_n\}$  is the standard simplex, the system (5.1) is linear and it has precisely  $n! \operatorname{Vol}_n(Q) = 1$  solution in  $(\mathbb{K}^*)^n$  if and only if no maximal minor of  $C \in \mathbb{K}^{n \times (n+1)}$  vanishes, in accordance with Cramer's rule for linear systems.

## 6. EXAMPLES

We conclude with a few examples illustrating the results of the previous section.

**Example 6.1.** Let  $\mathcal{A}_1 = \{(0, 0), (1, 2), (2, 1)\}$  and  $\mathcal{A}_2 = \{(2, 0), (0, 1), (1, 2)\}$  be individual supports, and  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$  the total support of a system. The Newton polytopes  $P_1 = \operatorname{conv} \mathcal{A}_1$ ,  $P_2 = \operatorname{conv} \mathcal{A}_2$ , and  $Q = \operatorname{conv} \mathcal{A}$  are depicted in Figure 1, where the vertices of  $P_1$  and  $P_2$  are labeled by  $\{1, 2, 3\}$  and  $\{4, 5, 2\}$ , respectively. We use the labeling in Figure 1 to order the columns of the augmented matrix

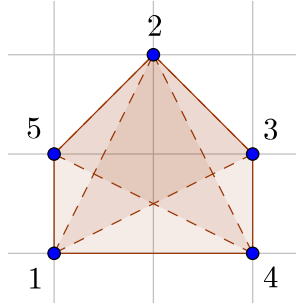


FIGURE 1. The mixed volume of the two triangles equals the volume of the pentagon.

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

A general system with these supports has the following coefficient matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 \\ 0 & c_{22} & 0 & c_{24} & c_{25} \end{pmatrix},$$

where  $c_{ij} \in \mathbb{K}$  are non-zero. One can see that each side of  $Q$  is touched by at least one of the  $P_i$  and, hence,  $V(P_1, P_2) = \text{Vol}_2(Q)$ , see Example 3.8. Also one can check that the rank conditions  $\text{rank } C_{\mathcal{F}} \geq \text{rank } \bar{A}_{\mathcal{F}}$  are satisfied for every face  $F$  of  $Q$ . (In fact, both ranks equal 2 when  $F$  is a side and 1 when  $F$  is a vertex.)

**Example 6.2.** Now we modify the previous example slightly, keeping  $\mathcal{A}_1$  the same and changing one of the points in  $\mathcal{A}_2$ , so  $\mathcal{A}_2 = \{(2, 0), (0, 1), (1, 1)\}$ , see Figure 2. The augmented exponent matrix and the coefficient matrix are as follows.

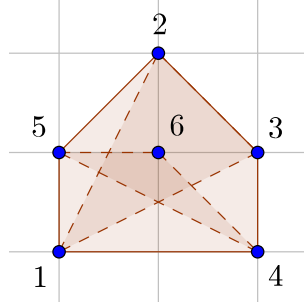


FIGURE 2. The mixed volume of the two triangles is less than the volume of the pentagon.

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{24} & c_{25} & c_{26} \end{pmatrix}.$$

This time the side of  $Q$  labeled by  $\mathcal{F} = \{2, 3\}$  is not touched by  $P_2$  and, hence,  $V(P_1, P_2) < \text{Vol}_2(Q)$ . Also, the rank condition for  $\mathcal{F} = \{2, 3\}$  fails:  $\text{rank } C_{\mathcal{F}} = 1$  and  $\text{rank } \bar{A}_{\mathcal{F}} = 2$ .

**Example 6.3.** Consider a system defined by the following augmented exponent matrix and coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}.$$

Here  $P_1 = P_2 = P_3 = Q$  which is a prism depicted in Figure 3. We label the vertices of  $Q$  using the order of the columns in  $\bar{A}$ . The submatrix of  $C$  corresponding to the edge  $F$  labeled  $\{5, 6\}$  has rank 1 which is less than  $\dim F + 1$ . Therefore the associated system has less than  $3! \text{Vol}_3(Q) = 3$  isolated solutions in  $(\mathbb{C}^*)^3$ . (In fact, it has two solutions.) In particular, this is a degenerate system.

In the following very particular situation, the rank condition (5.2) in Theorem 5.5 implies the non-degeneracy of the system.

**Remark 6.4.** Assume that  $P_1 = P_2 = \dots = P_n = Q$  with  $\dim Q = n$  and any proper face  $F$  of  $Q$  is a simplex which intersects  $\mathcal{A}$  only at its vertices. Assume furthermore that  $\text{rank } C_{\mathcal{F}} \geq \text{rank } \bar{A}_{\mathcal{F}}$  for any proper face  $F$  of  $Q$ . Then (5.1) is non-degenerate and thus has precisely  $n! \text{Vol}_n(Q)$  solutions in  $(\mathbb{K}^*)^n$  counted with

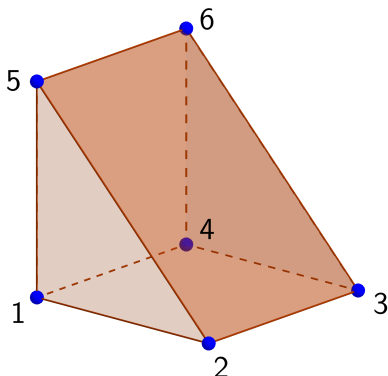


FIGURE 3. The Newton polytope of the system in Example 6.3.

multiplicity according to Theorem 5.1. Indeed, if  $F$  is a proper face of  $Q$ , then the corresponding restricted system has total support  $F \cap \mathcal{A}$ . If this restricted system is consistent, then there is a non-zero vector in the kernel of  $C_{\mathcal{F}}$  and thus  $\text{rank } C_{\mathcal{F}} < |F \cap \mathcal{A}| = 1 + \dim F$  which gives a contradiction.

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